

Exponential Convergence in L^p -Wasserstein Distance for Diffusion Processes without Uniformly Dissipative Drift

Dejun Luo^{a*} Jian Wang^{b†}

^aInstitute of Applied Mathematics, Academy of Mathematics and Systems Science,
Chinese Academy of Sciences, Beijing 100190, China

^bSchool of Mathematics and Computer Science, Fujian Normal University, Fuzhou 350007, China

Abstract

By adopting the coupling by reflection and choosing an auxiliary function which is convex near infinity, we establish the exponential convergence of diffusion semi-groups $(P_t)_{t \geq 0}$ with respect to the standard L^p -Wasserstein distance for all $p \in [1, \infty)$. In particular, we show that for the Itô stochastic differential equation

$$dX_t = dB_t + b(X_t) dt,$$

if the drift term b satisfies that for any $x, y \in \mathbb{R}^d$,

$$\langle b(x) - b(y), x - y \rangle \leq \begin{cases} K_1 |x - y|^2, & |x - y| \leq L; \\ -K_2 |x - y|^2, & |x - y| > L \end{cases}$$

holds with some positive constants K_1, K_2 and $L > 0$, then there is a constant $\lambda := \lambda(K_1, K_2, L) > 0$ such that for all $p \in [1, \infty)$, $t > 0$ and $x, y \in \mathbb{R}^d$,

$$W_p(\delta_x P_t, \delta_y P_t) \leq C e^{-\lambda t/p} \begin{cases} |x - y|^{1/p}, & \text{if } |x - y| \leq 1; \\ |x - y|, & \text{if } |x - y| > 1. \end{cases}$$

where $C := C(K_1, K_2, L, p)$ is a positive constant. This improves the main result in [13] where the exponential convergence is only proved for the L^1 -Wasserstein distance.

Keywords: Exponential convergence, L^p -Wasserstein distance, coupling by reflection, diffusion process

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†Email: jianwang@fjnu.edu.cn. Partly supported by NSFC (11201073 and 11522106), NSFFJ (2015J01003) and PNAIA (IRTL1206).

1 Introduction

In this paper we consider the following Itô stochastic differential equation

$$dX_t = \sigma dB_t + b(X_t) dt, \quad (1.1)$$

where $(B_t)_{t \geq 0}$ is a standard d -dimensional Brownian motion, $\sigma \in \mathbb{R}^{d \times d}$ is a non-degenerate constant matrix, and $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a Borel measurable vector field. Recently there are intensive studies on the existence of the unique strong solution to (1.1) with singular drift b . For example, if $\sigma = c \text{Id}$ for some constant $c \neq 0$ and b is bounded and Hölder continuous, Flandoli et al. [14] proved that (1.1) generates a unique flow of diffeomorphisms on \mathbb{R}^d . The results are recently extended by F.-Y. Wang in [28] to (infinite dimensional) stochastic differential equations with a nice multiplicative noise and a locally Dini continuous drift. From these results we see that when the diffusion coefficient σ is non-degenerate, quite weak conditions on b are sufficient to guarantee the well-posedness of (1.1), which will be assumed throughout this paper. Moreover, we assume the solution $(X_t)_{t \geq 0}$ has finite moments of all orders.

Denote by $(P_t)_{t \geq 0}$ the semigroup associated to (1.1). If the initial value X_0 is distributed as μ , then for any $t > 0$, the distribution of X_t is μP_t . We are concerned with the long-time behavior of P_t as t tends to ∞ , more exactly, the rate of convergence to equilibrium of $\delta_x P_t$ for any $x \in \mathbb{R}^d$. This problem is of fundamental importance in the study of Markov processes, and has been attacked by a large number of researchers in the literature. To the authors' knowledge, there are at least three approaches for obtaining quantitative ergodic properties. The first one is known as Harris' theorem which combines Lyapunov type conditions and the notion of small set, see [19, 20, 15] for systematic presentations. Recently this method is further developed in [16, 17, 4] with applications to stochastic partial differential equations (SPDE) and stochastic delay differential equations (SDDE). The second method employs functional inequalities to characterize the rate of convergence to equilibrium for $(P_t)_{t \geq 0}$. It is a classical result that for symmetric Markov processes the Poincaré inequality is equivalent to the exponential decay of the semigroup. More general functional inequalities were introduced in [26, 23, 27] to describe different convergence rates. It was shown in [1] that the two methods above can be linked together by Lyapunov–Poincaré inequalities. We also would like to mention that Bolley et al. [2] recently studied the exponential decay in the L^2 -Wasserstein distance W_2 via the so-called WJ inequality, by using the explicit formula for time derivative of W_2 along solutions to the Fokker–Planck equation (see [2, Theorem 2.1]). An application to the granular media was given in [3], yielding uniformly exponential convergence to equilibrium in the presence of non-convex interaction or confinement potentials.

Yet there is another approach for studying exponential convergence of the semigroup $(P_t)_{t \geq 0}$ corresponding to the SDE (1.1) considered in this paper, that is, the coupling method. If the drift vector field b fulfills certain dissipative properties, this latter method provides explicit rate of convergence to equilibrium in a straightforward way, see e.g. [5, 12] and the preprint [13]. The present work is motivated by [12, 13] where the author obtained exponential decay of $(P_t)_{t \geq 0}$ when the drift b is assumed to be only dissipative at infinity, see the introduction below for more details.

A good tool for measuring the deviation between probability distributions is the Wasserstein-type distances which are defined as follows. Let $\psi \in C^2([0, \infty))$ be a strictly

increasing function satisfying $\psi(0) = 0$. Given two probability measures μ and ν on \mathbb{R}^d , we define the following quantity

$$W_\psi(\mu, \nu) = \inf_{\Pi \in \mathcal{C}(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(|x - y|) d\Pi(x, y),$$

where $|\cdot|$ is the Euclidean norm and $\mathcal{C}(\mu, \nu)$ is the collection of measures on $\mathbb{R}^d \times \mathbb{R}^d$ having μ and ν as marginals. When ψ is concave, the above definition gives rise to a Wasserstein distance W_ψ in the space $\mathcal{P}(\mathbb{R}^d)$ of probability measures ν on \mathbb{R}^d such that $\int \psi(|z|) \nu(dz) < \infty$. If $\psi(r) = r$ for all $r \geq 0$, then W_ψ is the standard L^1 -Wasserstein distance (with respect to the Euclidean norm $|\cdot|$), which will be denoted by $W_1(\mu, \nu)$ throughout this paper. We are also concerned with the L^p -Wasserstein distance W_p for all $p \in [1, \infty)$, i.e.

$$W_p(\mu, \nu) = \inf_{\Pi \in \mathcal{C}(\mu, \nu)} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\Pi(x, y) \right)^{1/p}.$$

Equipped with W_p , the totality $\mathcal{P}_p(\mathbb{R}^d)$ of probability measures having finite moment of order p becomes a complete metric space.

In this paper, we shall establish the exponential convergence of the map $\mu \mapsto \mu P_t$ with respect to the L^p -Wasserstein distance W_p for all $p \geq 1$. We first recall some known results.

Theorem 1.1 (Uniformly dissipative case). *Suppose that $\sigma = \text{Id}$ and there exists a constant $K > 0$ such that*

$$\langle b(x) - b(y), x - y \rangle \leq -K|x - y|^2 \quad \text{for all } x, y \in \mathbb{R}^d. \quad (1.2)$$

Then, for any $p \geq 1$ and $t > 0$,

$$W_p(\mu P_t, \nu P_t) \leq e^{-Kt} W_p(\mu, \nu) \quad \text{for all } \mu, \nu \in \mathcal{P}_p(\mathbb{R}^d). \quad (1.3)$$

The proof of this result is quite straightforward, by simply using the synchronous coupling (also called the coupling of marching soldiers in [7, Example 2.16]), see e.g. [2, p.2432] for a short proof.

In applications, the so-called uniformly dissipative condition (1.2) is too strong. Indeed, it follows from [22, Theorem 1] or [2, Remark 3.6] (also see [5, Section 3.1.2, Theorem 1]) that (1.3) holds for any probability measures μ and ν if and only if (1.2) holds for all $x, y \in \mathbb{R}^d$. The first breakthrough to get rid of this restrictive condition was done recently by Eberle in [13], at the price of multiplying a constant $C \geq 1$ on the right hand side of (1.3). To state the main result in [13], we need the following notation which measures the dissipativity of the drift b :

$$\kappa(r) := \sup \left\{ \frac{\langle \sigma^{-1}(x - y), \sigma^{-1}(b(x) - b(y)) \rangle}{2|\sigma^{-1}(x - y)|} : x, y \in \mathbb{R}^d \text{ with } |\sigma^{-1}(x - y)| = r \right\}. \quad (1.4)$$

As in [13, (2.3)], we shall assume throughout the paper that

$$\int_0^s \kappa^+(r) dr < +\infty \quad \text{for all } s > 0.$$

This technical condition will be used in Section 2 to construct the auxiliary function.

Theorem 1.2 ([13, Corollary 2.3]). *Suppose that the vector field b is locally Lipschitz continuous, and there is a constant $c > 0$ such that*

$$\kappa(r) \leq -cr \quad \text{for all } r > 0 \text{ large enough.} \quad (1.5)$$

Then there exist positive constants $C, \lambda > 0$ such that for any $t > 0$ and $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$,

$$W_1(\mu P_t, \nu P_t) \leq C e^{-\lambda t} W_1(\mu, \nu).$$

In particular, when $\sigma = \text{Id}$, the condition (1.5) holds true if (1.2) is satisfied only for large $|x - y|$, that is,

$$\langle b(x) - b(y), x - y \rangle \leq -K|x - y|^2, \quad |x - y| \geq L$$

for some constant $L > 0$ large enough. Therefore, [13, Corollary 2.3] implies that the map $\mu \mapsto \mu P_t$ converges exponentially with respect to the standard L^1 -Wasserstein distance W_1 under locally non-dissipative drift, see [13, Example 1.1] for more details. The proof of [13, Corollary 2.3] is based on the coupling by reflection of diffusion processes and a carefully constructed concave function, cf. [13, Section 2]. A number of direct consequences are presented in [13, Section 2.2] which indicate that the convergence result as [13, Corollary 2.3] is extremely useful.

However, [13, Corollary 2.3] is not satisfactory in the sense that no information on the L^2 -Wasserstein distance W_2 is provided. This fact has also been noted in [5, Section 7.1, Remark 19], saying that “the reflection coupling cannot furnish some information on W_2 ”. Our main result of this paper shows that this is not the case.

Theorem 1.3. *Assume that there are constants $c > 0$ and $\eta \geq 0$ such that for all $r \geq \eta$, one has*

$$\kappa(r) \leq -cr. \quad (1.6)$$

For $\varepsilon \in (0, c)$, define

$$C_0(\varepsilon) = \max \left\{ \frac{2e^2}{\varepsilon} \left(1 + \frac{2}{\sqrt{\varepsilon}} \right) \sqrt{\frac{2}{c - \varepsilon}}, \frac{2 + \sqrt{\varepsilon}}{\varepsilon(1 - e^{-2})} \left[\frac{2\sqrt{2}e^2}{\sqrt{\varepsilon(c - \varepsilon)}} + \frac{1}{c - \varepsilon} \right] \right\}$$

and

$$\lambda = \frac{\min\{2, 2/\varepsilon\}}{C_0(\varepsilon)} \exp \left(-\frac{c}{2} \eta^2 - \int_0^\eta \kappa^+(s) \, ds \right).$$

Then for any $p > 1$, $t > 0$ and any $x, y \in \mathbb{R}^d$, it holds

$$W_p(\delta_x P_t, \delta_y P_t) \leq C e^{-\lambda t/p} \begin{cases} |x - y|^{1/p}, & \text{if } |x - y| \leq 1; \\ |x - y|, & \text{if } |x - y| > 1. \end{cases} \quad (1.7)$$

where $C := C(c, \eta, \varepsilon, p) > 0$ is a positive constant.

Theorem 1.3 above does provide new conditions on the drift term b such that the associated semigroup $(P_t)_{t \geq 0}$ is exponentially convergent with respect to the L^p -Wasserstein distance W_p for all $p \geq 1$. The reason that we can obtain the exponential convergence in W_p for all $p \geq 1$, not only W_1 , is due to our particular choice of the auxiliary function which is convex near infinity. It is designed by using Chen–Wang’s famous variational

formula for the principal eigenvalue of one-dimensional diffusion operator, see for instance [10] or [7, Section 3.4]. The reader can refer to [8] and the references therein for recent studies on this topic.

The assertion of Theorem 1.3 can be slightly strengthened if (1.6) is replaced by a stronger condition.

Theorem 1.4. *Assume that there are constants $c > 0$, $\eta > 0$ and $\theta > 1$ such that for all $r \geq \eta$, one has*

$$\kappa(r) \leq -cr^\theta. \quad (1.8)$$

Let λ be defined as in Theorem 1.3 with c replaced by $c\eta^{\theta-1}$. Then there is a positive constant C such that for all $t > 0$ and $x, y \in \mathbb{R}^d$, it holds

$$W_p(\delta_x P_t, \delta_y P_t) \leq Ce^{-\lambda t/p} \begin{cases} |x - y|^{1/p}, & \text{if } |x - y| \leq 1; \\ |x - y| \wedge \frac{1}{t \wedge 1}, & \text{if } |x - y| > 1. \end{cases} \quad (1.9)$$

The idea of the proof is to use synchronous coupling for large $|x - y|$ and the coupling by reflection for small $|x - y|$. For the latter part, we can directly use the result of Theorem 1.3, since (1.8) implies that (1.6) holds with $c\eta^{\theta-1}$ if $\eta > 0$.

Before presenting the consequences of Theorem 1.3, let us first make some comments and give some examples. In the beginning, we intended to generalize Eberle's results by assuming that *there is a constant $c > 0$ such that*

$$\kappa(r) \leq -c \quad \text{for all } r \text{ large enough.} \quad (1.10)$$

It turns out that under mild conditions on κ , the two conditions (1.5) and (1.10) are equivalent, up to changing the constants. More explicitly, we have

Proposition 1.5. *Assume that there are constants $c, r_0 > 0$ such that $\kappa(r_0) \leq -c$ and $\delta_0 := \sup_{0 \leq r \leq r_0} \kappa(r) < +\infty$. Then, the condition (1.5) holds with some other positive constant.*

This result indicates that if the function κ is locally bounded from above, then the following statements are equivalent:

- (i) there exist constants $c, r_0 > 0$ such that $\kappa(r_0) \leq -c$;
- (ii) there exist constants $c > 0$ and $\theta \leq 1$ such that $\kappa(r) \leq -cr^\theta$ for $r > 0$ large enough;
- (iii) there exists a constant $c > 0$ such that $\kappa(r) \leq -cr$ for $r > 0$ large enough.

Note that, none of the above condition is equivalent to that there exist constants $c > 0$ and $\theta > 1$ such that for $r > 0$ large enough, $\kappa(r) \leq -cr^\theta$, see e.g. $b(x) = -x$ for all $x \in \mathbb{R}^d$. Compared to (1.5), the seemingly much weaker condition (i), i.e. *there exist two constants $c, r_0 > 0$ such that $\kappa(r_0) \leq -c$* , has the obvious advantage of being easily verifiable. Thus we shall sometimes use this formulation in the sequel.

The equivalence stated above also indicates that Theorem 1.3 is sharp in some situation, as shown by the next example.

Example 1.6. *Assume that $\sigma = \text{Id}$ and $b(x) = \nabla V(x)$ with $V(x) = -(1 + |x|^2)^{\delta/2}$ for some $\delta \in (0, 2)$. Then, we have the following statements.*

- (1) If $\delta \in (0, 1)$, then $\kappa(r) \geq 0$ for all r large enough, and the inequality (1.7) does not hold for any positive constants C and λ with $p = 1$.
- (2) If $\delta \in [1, 2)$, then $\kappa(r) = 0$ for all $r \geq 0$, and so for all $x, y \in \mathbb{R}^d$ and $t > 0$,

$$W_1(\delta_x P_t, \delta_y P_t) \leq |x - y|. \quad (1.11)$$

On the other hand, it holds that

$$d_{TV}(\delta_x P_t, \delta_y P_t) \leq \sqrt{\frac{2}{\pi t}} |x - y| \quad \text{for all } t \geq 0 \text{ and } x, y \in \mathbb{R}^d, \quad (1.12)$$

where d_{TV} is the total variation distance between probability measures.

To show the power of Theorem 1.4, we consider the following example which yields the exponential convergence of the semigroup $(P_t)_{t \geq 0}$ with respect to the L^p -Wasserstein distance W_p ($p > 2$) for super-convex potentials. The assertion below improves the results mentioned in [5, Section 6.1].

Example 1.7. Let $\sigma = \text{Id}$ and $b(x) = \nabla V(x)$ with $V(x) = -|x|^{2\alpha}$ and $\alpha > 1$. It follows from [5, Section 6, Example 1] that there is a constant $c > 0$ such that for all $r > 0$,

$$\kappa(r) \leq -cr^{2\alpha-1}. \quad (1.13)$$

Then, according to Theorem 1.4, the associated semigroup $(P_t)_{t \geq 0}$ converges exponentially with respect to the L^p -Wasserstein distance W_p for any $p \geq 1$. More explicitly, there is a constant $\lambda := \lambda(\alpha) > 0$ such that for any $p \geq 1$, $x, y \in \mathbb{R}^d$ and $t > 0$,

$$W_p(\delta_x P_t, \delta_y P_t) \leq C e^{-\lambda t} \left[|x - y|^{1/p} \mathbf{1}_{\{|x-y| \leq 1\}} + \left(|x - y| \wedge \frac{1}{t \wedge 1} \right) \mathbf{1}_{\{|x-y| \geq 1\}} \right]$$

holds with some constant $C := C(\alpha, p) > 0$.

Note that (1.13) implies that for all $x, y \in \mathbb{R}^d$,

$$\langle b(x) - b(y), x - y \rangle \leq -2c|x - y|^{2\alpha}.$$

Therefore, the uniformly dissipative condition (1.2) fails when $x, y \in \mathbb{R}^d$ are sufficiently close to each other. That is, one cannot deduce directly from Theorem 1.1 the exponential convergence with respect to the L^p -Wasserstein distance W_p ($p \geq 1$).

As applications of Theorem 1.3, we consider the existence and uniqueness of the invariant probability measure, and also the exponential convergence of the semigroup with respect to the L^p -Wasserstein distance W_p . For $p \in (1, \infty)$, we define

$$\phi_p(r) = \begin{cases} r^{1/p}, & \text{if } r < p^{-p/(p-1)}; \\ r - p^{-p/(p-1)} + p^{-1/(p-1)}, & \text{if } r \geq p^{-p/(p-1)}. \end{cases}$$

Finally, let $\phi_1(r) = r$ for all $r \geq 0$. Then for all $p \in [1, \infty)$, ϕ_p is a concave C^1 -function on \mathbb{R}_+ , thus W_{ϕ_p} is a well defined distance on $\mathcal{P}_p(\mathbb{R}^d)$. Moreover,

$$r \vee r^{1/p} \leq \phi_p(r) \leq r + r^{1/p} \leq 2(r \vee r^{1/p}) \quad \text{for all } r \geq 0, p \in [1, \infty).$$

Corollary 1.8. *Suppose that the drift term b is locally bounded on \mathbb{R}^d , and (1.6) holds for all $r > 0$ large enough with some constant $c > 0$. Then, there exists a unique invariant probability measure $\mu \in \cap_{p \geq 1} \mathcal{P}_p(\mathbb{R}^d)$, such that there is a constant $\lambda := \lambda(c) > 0$ such that for all $p \in [1, \infty)$ and for any probability measure $\nu \in \mathcal{P}_p(\mathbb{R}^d)$,*

$$W_p(\nu P_t, \mu) \leq C e^{-\lambda t} W_{\phi_p}(\nu, \mu), \quad t \geq 0$$

holds with some positive constant $C := C(c, p)$.

Remark 1.9. (1) Under the assumptions of Corollary 1.8, it is easy to establish the following Foster-Lyapunov type conditions:

$$L\phi(x) \leq -c_1\phi(x) + c_2, \quad x \in \mathbb{R}^d,$$

where L is the generator of the underlying diffusion process, $\phi(x) = |x|^2$ and c_1, c_2 are two positive constants. Due to the existence and uniqueness of the invariant probability measure, we know that the diffusion process exponentially converges to the unique invariant probability measure μ with respect to the total variation distance. That is, there are a constant $\theta > 0$ and a measurable function $C(x)$ such that for all $x \in \mathbb{R}^d$ and $t > 0$,

$$d_{TV}(P(t, \cdot), \pi) \leq C(x)e^{-\theta t},$$

where $P(t, \cdot)$ is the associated transition probability.

- (2) As was pointed by the referee, the conclusion of Corollary 1.8 for $p = 1$ also could be deduced from Theorem 1.4 and the Foster-Lyapunov type condition above, by Harris's theorem for the exponential convergence to the invariant measure in the Wasserstein metric. See [17, Theorem 4.8] and [4, Theorem 2.4] for more details.

The following statement is concerned with symmetric diffusion processes. Though we believe the assertion below is known (see e.g. [25, Corollary 1.4]), we stress the relation between the exponential convergence with respect to L^1 -Wasserstein distance W_1 and that with respect to the L^2 -norm, which is equivalent to the Poincaré inequality.

Corollary 1.10. *Let U be a C^2 -potential defined on \mathbb{R}^d such that its Hessian matrix $\text{Hess}(U) \geq -K$ for some $K > 0$. Assume that $\mu(dx) = e^{-U(x)} dx$ is a probability measure on \mathbb{R}^d . If there exists a constant $L > 0$ such that*

$$\inf_{|x-y|=L} \langle \nabla U(x) - \nabla U(y), x - y \rangle > 0, \quad (1.14)$$

then μ satisfies the Poincaré inequality, i.e.

$$\mu(f^2) - \mu(f)^2 \leq C \int |\nabla f(x)|^2 d\mu, \quad f \in C_c^2(\mathbb{R}^d) \quad (1.15)$$

holds for some constant $C > 0$.

2 Preliminaries

2.1 Coupling by Reflection

Similar to the main result in [13], the proof of Theorem 1.3 is based on the reflection coupling of Brownian motion, which was introduced by Lindvall and Rogers [18] and developed by Chen and Li [9]. First, we give a brief introduction of the coupling by reflection. Together with (1.1), we also consider

$$dY_t = \sigma(\text{Id} - 2e_t e_t^*) dB_t + b(Y_t) dt, \quad t < T, \quad (2.1)$$

where $\text{Id} \in \mathbb{R}^{d \times d}$ is the identity matrix,

$$e_t = \frac{\sigma^{-1}(X_t - Y_t)}{|\sigma^{-1}(X_t - Y_t)|}$$

and $T = \inf\{t > 0 : X_t = Y_t\}$ is the coupling time. For $t \geq T$, we shall set $Y_t = X_t$. Then, the process $(X_t, Y_t)_{t \geq 0}$ is called the coupling by reflection of $(X_t)_{t \geq 0}$. Under our assumptions, the reflection coupling $(X_t, Y_t)_{t \geq 0}$ can be realized as a non-explosive diffusion process in \mathbb{R}^{2d} . The difference process $(Z_t)_{t \geq 0} = (X_t - Y_t)_{t \geq 0}$ satisfies

$$dZ_t = \frac{2Z_t}{|\sigma^{-1}Z_t|} dW_t + (b(X_t) - b(Y_t)) dt, \quad t < T, \quad (2.2)$$

where $(W_t)_{0 \leq t < T}$ is a one dimensional Brownian motion expressed by $W_t = \int_0^t \langle e_s, dB_s \rangle$.

Next, by Itô's formula and (2.2), for $t < T$,

$$\begin{aligned} d(|\sigma^{-1}Z_t|^2) &= 2\langle \sigma^{-1}Z_t, \sigma^{-1}dZ_t \rangle + \langle \sigma^{-1}dZ_t, \sigma^{-1}dZ_t \rangle \\ &= 4|\sigma^{-1}Z_t| dW_t + 2\langle \sigma^{-1}Z_t, \sigma^{-1}(b(X_t) - b(Y_t)) \rangle dt + 4 dt. \end{aligned}$$

Denote by $r_t = |\sigma^{-1}Z_t|$. Then

$$\begin{aligned} dr_t &= \frac{1}{2r_t} dr_t^2 - \frac{1}{8r_t^3} dr_t^2 \cdot dr_t^2 \\ &= 2 dW_t + \frac{1}{r_t} \langle \sigma^{-1}Z_t, \sigma^{-1}(b(X_t) - b(Y_t)) \rangle dt. \end{aligned} \quad (2.3)$$

2.2 Auxiliary Function

For any $\varepsilon \in (0, c)$, let $\psi \in C^2([0, \infty))$ be the following strictly increasing function

$$\psi(r) = \int_0^r \exp\left(-\int_0^s \kappa^*(v) dv\right) \left\{ \int_s^\infty \exp\left(\int_0^u [\kappa^*(v) + \varepsilon v] dv\right) du \right\} ds, \quad (2.4)$$

where

$$\kappa^*(r) = \begin{cases} \kappa^+(r), & \text{if } 0 < r \leq \eta; \\ -cr, & \text{if } r > \eta. \end{cases}$$

Remark 2.1. The definition of ψ seems a little bit strange at first sight, indeed, it is motivated by Chen–Wang’s variational formula for principal eigenvalue of one-dimensional diffusion operator

$$Lf(r) = f''(r) + b(r)f'(r)$$

on $[0, \infty)$. One key to this famous formula is the following elegant mimic eigenfunction g associated with the first eigenvalue (see [7, pp. 52–53] for a heuristic argument):

$$g(r) = \int_0^r e^{-C(s)} ds \int_s^\infty h(u) e^{C(u)} du,$$

where

$$C(s) = \int_0^s b(u) du, \quad h(s) = \left(\int_0^r e^{-C(s)} ds \right)^{1/2}.$$

Now, let $b(r) = -cr$ in the definition of the function g above. Then we have

$$g(r) \asymp \psi(r) \quad \text{as } r \rightarrow \infty$$

for some proper choice of the constant ε in the definition of ψ .

On the one hand, it is easy to see that for any $r > 0$,

$$\begin{aligned} \psi'(r) &= \exp \left(- \int_0^r \kappa^*(v) dv \right) \int_r^\infty \exp \left(\int_0^u [\kappa^*(v) + \varepsilon v] dv \right) du, \\ \psi''(r) &= -\kappa^*(r)\psi'(r) - \exp(\varepsilon r^2/2), \end{aligned}$$

and so

$$\psi''(r) + \kappa(r)\psi'(r) \leq -\exp(\varepsilon r^2/2), \quad r \geq 0. \quad (2.5)$$

On the other hand, for all $r > 0$,

$$\kappa^*(r) = [\kappa^+(r) + cr] \mathbf{1}_{[0, \eta]}(r) - cr.$$

Thus, for all $r > 0$,

$$\begin{aligned} \psi(r) &= \int_0^r \exp \left(\frac{c}{2}s^2 - \int_0^{\eta \wedge s} [\kappa^+(v) + cv] dv \right) \\ &\quad \times \left\{ \int_s^\infty \exp \left(-\frac{c-\varepsilon}{2}u^2 + \int_0^{u \wedge \eta} [\kappa^+(v) + cv] dv \right) du \right\} ds \\ &\leq \exp \left(\int_0^\eta [\kappa^+(v) + cv] dv \right) \int_0^r e^{cs^2/2} \left(\int_s^\infty e^{-(c-\varepsilon)u^2/2} du \right) ds. \end{aligned} \quad (2.6)$$

Define

$$\psi_1(r) = \int_0^r e^{cs^2/2} \left(\int_s^\infty e^{-(c-\varepsilon)u^2/2} du \right) ds, \quad \psi_2(r) = \frac{e^{\varepsilon r^2/2} - 1}{r(1+r)}.$$

We first show that $\psi_1(r)$ and $\psi_2(r)$ are comparable.

Lemma 2.2. *There exist two constants $C_0 := C_0(\varepsilon)$ and $\hat{C}_0 := \hat{C}_0(\varepsilon) > 0$ such that for all $r \geq 0$,*

$$\hat{C}_0 \psi_2(r) \leq \psi_1(r) \leq C_0 \psi_2(r).$$

Proof. Note that $r(1+r) \sim r$ as $r \rightarrow 0$ and $r(1+r) \sim r^2$ as $r \rightarrow \infty$, where \sim means the two quantities are of the same order. By L'Hôpital's law,

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{\psi_1(r)}{\psi_2(r)} &= \lim_{r \rightarrow 0} \frac{e^{cr^2/2} \int_r^\infty e^{-(c-\varepsilon)u^2/2} du}{-r^{-2}(e^{\varepsilon r^2/2} - 1) + r^{-1}e^{\varepsilon r^2/2}\varepsilon r} \\ &= \frac{2}{\varepsilon} \int_0^\infty e^{-(c-\varepsilon)u^2/2} du \\ &= \frac{2}{\varepsilon} \sqrt{\frac{\pi}{2(c-\varepsilon)}}. \end{aligned}$$

Next, using L'Hôpital's law twice,

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{\psi_1(r)}{\psi_2(r)} &= \lim_{r \rightarrow \infty} \frac{\psi_1(r)}{r^{-2}e^{\varepsilon r^2/2}} \\ &= \lim_{r \rightarrow \infty} \frac{e^{cr^2/2} \int_r^\infty e^{-(c-\varepsilon)u^2/2} du}{e^{\varepsilon r^2/2}[-2r^{-3} + \varepsilon r^{-1}]} \\ &= \lim_{r \rightarrow \infty} \frac{\int_r^\infty e^{-(c-\varepsilon)u^2/2} du}{e^{-(c-\varepsilon)r^2/2} r^{-2}[-2r^{-1} + \varepsilon r]} \\ &= - \lim_{r \rightarrow \infty} \frac{1}{h(r)}, \end{aligned}$$

where

$$h(r) = -(c-\varepsilon)r^{-1}(-2r^{-1} + \varepsilon r) - 2r^{-3}(-2r^{-1} + \varepsilon r) + r^{-2}(2r^{-2} + \varepsilon).$$

Thus,

$$\lim_{r \rightarrow \infty} \frac{\psi_1(r)}{\psi_2(r)} = \frac{1}{\varepsilon(c-\varepsilon)}.$$

Therefore, the required assertion follows from the two limits above. \square

By (2.6) and Lemma 2.2, we have

$$\psi(r) \leq C_0 \exp \left(\int_0^\eta [\kappa^+(v) + cv] dv \right) \frac{e^{\varepsilon r^2/2} - 1}{r(1+r)}$$

and

$$\psi(r) \geq \hat{C}_0 \exp \left(- \int_0^\eta [\kappa^+(v) + cv] dv \right) \frac{e^{\varepsilon r^2/2} - 1}{r(1+r)}.$$

Roughly speaking, the above two inequalities imply that the auxiliary function ψ behaves like $c'r$ for small r , and grows exponentially fast as $e^{c''r^2}$ for large r . Hence, the function $\psi(r)$ can be used to control the function r^p with $p \geq 1$. More explicitly, we have

Corollary 2.3. *There is a constant $C_1 > 1$ such that for all $r \geq 0$,*

$$C_1^{-1}[r \vee (e^{\varepsilon r^2/2} - 1)] \leq \psi(r) \leq C_1[r \vee (e^{\varepsilon r^2/2} - 1)]. \quad (2.7)$$

Consequently, for any $p \geq 1$, there is a constant $C_2 = C_2(p, \varepsilon) > 0$ such that for all $r \geq 0$,

$$r^p \leq C_2 \psi(r). \quad (2.8)$$

Furthermore, we can give an explicit estimate to the constant C_0 in Lemma 2.2, which will be used in the exponential convergence rate.

Lemma 2.4. *The constant C_0 in Lemma 2.2 has the following expression:*

$$C_0 = \max \left\{ \frac{2e^2}{\varepsilon} \left(1 + \frac{2}{\sqrt{\varepsilon}} \right) \sqrt{\frac{2}{c-\varepsilon}}, \frac{2+\sqrt{\varepsilon}}{\varepsilon(1-e^{-2})} \left[\frac{2\sqrt{2}e^2}{\sqrt{\varepsilon(c-\varepsilon)}} + \frac{1}{c-\varepsilon} \right] \right\}.$$

Proof. In order to estimate $\psi_1(r)$, we need the following inequality on the tail of standard Gaussian distribution (e.g. see [11, (3)]):

$$1 - \Phi(r) \leq \frac{2\phi(r)}{\sqrt{2+r^2}+r} \quad \text{for all } r > 0,$$

where $\Phi(r)$ and $\phi(r)$ are respectively the distribution and density function of the standard Gaussian distribution $N(0, 1)$. Consequently, for $s > 0$,

$$\int_s^\infty e^{-(c-\varepsilon)u^2/2} du = \frac{1}{\sqrt{c-\varepsilon}} \int_{\sqrt{c-\varepsilon}s}^\infty e^{-v^2/2} dv \leq \frac{1}{\sqrt{c-\varepsilon}} \cdot \frac{2e^{-(c-\varepsilon)s^2/2}}{\sqrt{2+(c-\varepsilon)s^2}+\sqrt{c-\varepsilon}s}.$$

Substituting this estimate into the expression of ψ_1 leads to

$$\psi_1(r) \leq \frac{2}{\sqrt{c-\varepsilon}} \int_0^r \frac{e^{\varepsilon s^2/2}}{\sqrt{2+(c-\varepsilon)s^2}+\sqrt{c-\varepsilon}s} ds =: \frac{2}{\sqrt{c-\varepsilon}} \tilde{\psi}_1(r). \quad (2.9)$$

Next, we consider two cases. (i) If $r \leq 2/\sqrt{\varepsilon}$, then

$$\tilde{\psi}_1(r) \leq \int_0^r \frac{e^2}{\sqrt{2}} ds = \frac{e^2}{\sqrt{2}} r.$$

(ii) If $r > 2/\sqrt{\varepsilon}$, then

$$\begin{aligned} \tilde{\psi}_1(r) &= \left(\int_0^{2/\sqrt{\varepsilon}} + \int_{2/\sqrt{\varepsilon}}^r \right) \frac{e^{\varepsilon s^2/2}}{\sqrt{2+(c-\varepsilon)s^2}+\sqrt{c-\varepsilon}s} ds \\ &\leq e^2 \sqrt{\frac{2}{\varepsilon}} + \frac{1}{2\sqrt{c-\varepsilon}} \int_{2/\sqrt{\varepsilon}}^r \frac{e^{\varepsilon s^2/2}}{s} ds. \end{aligned} \quad (2.10)$$

By the integration by parts formula,

$$\begin{aligned} \int_{2/\sqrt{\varepsilon}}^r \frac{e^{\varepsilon s^2/2}}{s} ds &= \frac{1}{\varepsilon} \int_{2/\sqrt{\varepsilon}}^r \frac{d(e^{\varepsilon s^2/2})}{s^2} \\ &= \frac{1}{\varepsilon} \left[\frac{e^{\varepsilon r^2/2}}{r^2} - \frac{\varepsilon}{4} e^2 + \int_{2/\sqrt{\varepsilon}}^r \frac{e^{\varepsilon s^2/2}}{s^3} ds \right] \\ &\leq \frac{e^{\varepsilon r^2/2}}{\varepsilon r^2} + \frac{1}{2} \int_{2/\sqrt{\varepsilon}}^r \frac{e^{\varepsilon s^2/2}}{s} ds. \end{aligned}$$

Therefore,

$$\int_{2/\sqrt{\varepsilon}}^r \frac{e^{\varepsilon s^2/2}}{s} ds \leq \frac{2e^{\varepsilon r^2/2}}{\varepsilon r^2}.$$

Substituting this estimate into (2.10) yields

$$\tilde{\psi}_1(r) \leq e^2 \sqrt{\frac{2}{\varepsilon}} + \frac{e^{\varepsilon r^2/2}}{\varepsilon \sqrt{c-\varepsilon} r^2}.$$

Summarizing the above two cases and using (2.9), we obtain

$$\psi_1(r) \leq \begin{cases} \sqrt{\frac{2}{c-\varepsilon}} e^2 r, & \text{if } r \leq \frac{2}{\sqrt{\varepsilon}}; \\ \frac{2\sqrt{2}e^2}{\sqrt{\varepsilon(c-\varepsilon)}} + \frac{2e^{\varepsilon r^2/2}}{\varepsilon(c-\varepsilon)r^2}, & \text{if } r > \frac{2}{\sqrt{\varepsilon}}. \end{cases} \quad (2.11)$$

Furthermore, since

$$e^{\varepsilon r^2/2} - 1 \geq \varepsilon r^2/2, \quad r \geq 0,$$

it is easy to show that for all $r \in [0, 2/\sqrt{\varepsilon}]$, it holds

$$\sqrt{\frac{2}{c-\varepsilon}} e^2 r \leq \frac{C_3}{r(1+r)} (e^{\varepsilon r^2/2} - 1), \quad (2.12)$$

where

$$C_3 = \frac{2e^2}{\varepsilon} \left(1 + \frac{2}{\sqrt{\varepsilon}}\right) \sqrt{\frac{2}{c-\varepsilon}}.$$

On the other hand, for $r > 2/\sqrt{\varepsilon}$, we have

$$\begin{aligned} r(1+r) \left[\frac{2\sqrt{2}e^2}{\sqrt{\varepsilon(c-\varepsilon)}} + \frac{2e^{\varepsilon r^2/2}}{\varepsilon(c-\varepsilon)r^2} \right] &= \left(1 + \frac{1}{r}\right) \left[\frac{2\sqrt{2}e^2}{\sqrt{\varepsilon(c-\varepsilon)}} r^2 + \frac{2e^{\varepsilon r^2/2}}{\varepsilon(c-\varepsilon)} \right] \\ &\leq \frac{2+\sqrt{\varepsilon}}{\varepsilon} \left[\frac{2\sqrt{2}e^2}{\sqrt{\varepsilon(c-\varepsilon)}} + \frac{1}{c-\varepsilon} \right] e^{\varepsilon r^2/2} \end{aligned}$$

and

$$e^{\varepsilon r^2/2} - 1 \geq (1 - e^{-2}) e^{\varepsilon r^2/2}.$$

Combining the above two inequalities, we deduce that for all $r > 2/\sqrt{\varepsilon}$,

$$\frac{2\sqrt{2}e^2}{\sqrt{\varepsilon(c-\varepsilon)}} + \frac{2e^{\varepsilon r^2/2}}{\varepsilon(c-\varepsilon)r^2} \leq \frac{C_4}{r(1+r)} (e^{\varepsilon r^2/2} - 1). \quad (2.13)$$

with

$$C_4 = \frac{2+\sqrt{\varepsilon}}{\varepsilon(1-e^{-2})} \left[\frac{2\sqrt{2}e^2}{\sqrt{\varepsilon(c-\varepsilon)}} + \frac{1}{c-\varepsilon} \right].$$

Having the two inequalities (2.12) and (2.13) in hand, and using (2.11), we complete the proof. \square

We also need the following simple result.

Lemma 2.5. For all $r > 0$,

$$e^{\varepsilon r^2/2} \geq \bar{C}_0 \psi_2(r), \quad (2.14)$$

where

$$\bar{C}_0 = \min\{2, 2/\varepsilon\}.$$

Proof. It is obvious that if $r \geq 1$, then

$$\psi_2(r) = \frac{e^{\varepsilon r^2/2} - 1}{r(1+r)} \leq \frac{e^{\varepsilon r^2/2}}{2}.$$

If $0 \leq r \leq 1$, then, using the fact that

$$e^r - 1 \leq r e^r, \quad r \geq 0,$$

we have

$$\psi_2(r) = \frac{e^{\varepsilon r^2/2} - 1}{r(1+r)} \leq \frac{\varepsilon r e^{\varepsilon r^2/2}}{2} \leq \frac{\varepsilon e^{\varepsilon r^2/2}}{2}.$$

The required assertion follows immediately from these two conclusions. \square

Finally, we present a consequence of all the previous results in this part.

Corollary 2.6. Let ψ be the function defined by (2.4), and λ be the constant in Theorem 1.3. Then, for all $r > 0$,

$$\psi''(r) + \kappa(r)\psi'(r) \leq -\lambda\psi(r). \quad (2.15)$$

Proof. By (2.5), we deduce from Lemmas 2.4 and 2.5 that

$$\begin{aligned} \psi''(r) + \kappa(r)\psi'(r) &\leq -e^{\varepsilon r^2/2} \leq -\bar{C}_0 \psi_2(r) \leq -\frac{\bar{C}_0}{C_0} \psi_1(r) \\ &\leq -\frac{\bar{C}_0}{C_0} \exp\left(-\frac{c}{2}\eta^2 - \int_0^\eta \kappa^+(s) ds\right) \psi(r), \end{aligned}$$

where the last inequality follows from (2.6). This, along with the definition of the constant λ in Theorem 1.3, yields the required assertion. \square

3 Proofs of Theorems and Proposition

We first give the

Proof of Theorem 1.3. Recall that we assume the solution $(X_t)_{t \geq 0}$ to (1.1) has finite moments of all orders. In particular, the left hand side of (1.7) is finite for any $x, y \in \mathbb{R}^d$ and $t > 0$.

Let ψ be the function defined by (2.4). According to (2.3) and Itô's formula, it holds that

$$\begin{aligned} d\psi(r_t) &= 2\psi'(r_t) dW_t + 2\left[\psi''(r_t) + \frac{\psi'(r_t)}{2r_t} \langle \sigma^{-1} Z_t, \sigma^{-1}(b(X_t) - b(Y_t)) \rangle\right] dt \\ &\leq 2\psi'(r_t) dW_t + 2[\psi''(r_t) + \kappa(r_t)\psi'(r_t)] dt. \end{aligned}$$

Combining this inequality with (2.15), we obtain

$$d\psi(r_t) \leq 2\psi'(r_t) dW_t - \lambda\psi(r_t) dt, \quad (3.1)$$

where λ is the constant in Theorem 1.3.

For $n \geq 1$, define the stopping time

$$T_n = \inf\{t > 0 : r_t \notin [1/n, n]\}.$$

Then, for $t \leq T_n$, the inequality (3.1) yields

$$d[e^{\lambda t}\psi(r_t)] \leq 2e^{\lambda t}\psi'(r_t) dW_t.$$

Therefore,

$$e^{\lambda(t \wedge T_n)}\psi(r_{t \wedge T_n}) \leq \psi(r_0) + 2 \int_0^{t \wedge T_n} e^{\lambda s}\psi'(r_s) dW_s.$$

Taking expectation in the both hand sides of the inequality above leads to

$$\mathbb{E}[e^{\lambda(t \wedge T_n)}\psi(r_{t \wedge T_n})] \leq \psi(r_0).$$

Since the coupling process $(X_t, Y_t)_{t \geq 0}$ is non-explosive, we have $T_n \uparrow T$ a.s. as $n \rightarrow \infty$, where T is the coupling time. Thus by Fatou's lemma, letting $n \rightarrow \infty$ in the above inequality gives us

$$\mathbb{E}[e^{\lambda(t \wedge T)}\psi(r_{t \wedge T})] \leq \psi(r_0). \quad (3.2)$$

Thanks to our convention that $Y_t = X_t$ for $t \geq T$, we have $r_t = 0$ for all $t \geq T$. Therefore,

$$\mathbb{E}[e^{\lambda(t \wedge T)}\psi(r_{t \wedge T})] = \mathbb{E}[e^{\lambda t}\psi(r_t)\mathbf{1}_{\{T > t\}}] = \mathbb{E}[e^{\lambda t}\psi(r_t)].$$

Combining this with (3.2), we arrive at

$$\mathbb{E}\psi(r_t) \leq \psi(r_0)e^{-\lambda t}.$$

That is,

$$\mathbb{E}\psi(|\sigma^{-1}(X_t - Y_t)|) \leq \psi(|\sigma^{-1}(x - y)|)e^{-\lambda t}. \quad (3.3)$$

If $|\sigma^{-1}(x - y)| \leq \eta$, then for any $p \geq 1$ and any $t > 0$, we deduce from (2.8), (3.3) and (2.7) that

$$\mathbb{E}(|\sigma^{-1}(X_t - Y_t)|^p) \leq C_2 \mathbb{E}\psi(|\sigma^{-1}(X_t - Y_t)|) \leq C_5 e^{-\lambda t} |\sigma^{-1}(x - y)|. \quad (3.4)$$

It is clear that

$$C_6^{-1}|z| \leq |\sigma^{-1}z| \leq C_6|z|, \quad z \in \mathbb{R}^d$$

for some constant $C_6 > 1$. Therefore, if $|x - y| \leq \eta/C_6$, then for any $p \geq 1$ there is a constant $C_7 > 0$ such that

$$\mathbb{E}|X_t - Y_t|^p \leq C_7 e^{-\lambda t} |x - y|, \quad t > 0,$$

which implies that for any $p \geq 1$ and any $x, y \in \mathbb{R}^d$ with $|x - y| \leq \eta/C_6$,

$$W_p(\delta_x P_t, \delta_y P_t) \leq C_7^{1/p} e^{-\lambda t/p} |x - y|^{1/p}. \quad (3.5)$$

Now for any $x, y \in \mathbb{R}^d$ with $|x - y| > \eta/C_6$, take $n := \lceil C_6|x - y|/\eta \rceil + 1 \geq 2$. We have

$$\frac{n}{2} \leq n - 1 \leq \frac{C_6|x - y|}{\eta} \leq n. \quad (3.6)$$

Set $x_i = x + i(y - x)/n$ for $i = 0, 1, \dots, n$. Then $x_0 = x$ and $x_n = y$; moreover, (3.6) implies $|x_{i-1} - x_i| = |x - y|/n \leq \eta/C_6$ for all $i = 1, 2, \dots, n$. Therefore, for all $p \geq 1$, by (3.5),

$$\begin{aligned} W_p(\delta_x P_t, \delta_y P_t) &\leq \sum_{i=1}^n W_p(\delta_{x_{i-1}} P_t, \delta_{x_i} P_t) \\ &\leq C_7^{1/p} e^{-\lambda t/p} \sum_{i=1}^n |x_{i-1} - x_i|^{1/p} \\ &\leq C_7^{1/p} e^{-\lambda t/p} n(\eta/C_6)^{1/p} \\ &\leq C_8 e^{-\lambda t/p} |x - y|, \end{aligned}$$

where in the last inequality we have used $n \leq 2C_6|x - y|/\eta$. The proof of Theorem 1.3 is completed. \square

Next, we turn to the

Proof of Theorem 1.4. Since we assume $\eta > 0$, condition (1.8) implies that (1.6) holds with c replaced by $c\eta^{\theta-1}$. Then, we can directly apply the assertion of Theorem 1.3 to conclude that there exists a constant λ (which is given in Theorem 1.3 with c replaced by $c\eta^{\theta-1}$) such that for all $p \geq 1$ and $x, y \in \mathbb{R}^d$

$$W_p(\delta_x P_t, \delta_y P_t) \leq C e^{-\lambda t/p} (|x - y|^{1/p} \vee |x - y|). \quad (3.7)$$

To complete the proof, we only need to consider the case that $x, y \in \mathbb{R}^d$ with $|\sigma^{-1}(x - y)| > \eta$ and $t > 0$ large enough. For this, we use both the synchronous coupling and the coupling by reflection defined by (2.1). In details, with (1.1), we now consider

$$dY_t = \begin{cases} \sigma dB_t + b(Y_t) dt, & 0 \leq t < T_\eta, \\ \sigma(\text{Id} - 2e_t e_t^*) dB_t + b(Y_t) dt, & T_\eta \leq t < T, \end{cases}$$

where

$$T_\eta = \inf\{t > 0 : |\sigma^{-1}(X_t - Y_t)| = \eta\}$$

and $T = \inf\{t > 0 : X_t = Y_t\}$ is the coupling time. For $t \geq T$, we still set $Y_t = X_t$. Therefore, the difference process $(Z_t)_{t \geq 0} = (X_t - Y_t)_{t \geq 0}$ satisfies

$$dZ_t = (b(X_t) - b(Y_t)) dt, \quad t < T_\eta.$$

As a result,

$$d|\sigma^{-1}Z_t|^2 = 2\langle \sigma^{-1}Z_t, \sigma^{-1}(b(X_t) - b(Y_t)) \rangle dt.$$

Still denoting by $r_t = |\sigma^{-1}Z_t|$, we get

$$dr_t \leq 2\kappa(r_t) dt \leq -2cr_t^\theta dt, \quad t < T_\eta,$$

which implies that

$$T_\eta \leq \frac{1}{2c(1-\theta)} (|\sigma^{-1}(x-y)|^{1-\theta} - \eta^{1-\theta}) \leq \frac{\eta^{1-\theta}}{2c(\theta-1)} =: t_0 \quad (3.8)$$

since $\theta > 1$.

Therefore, for any $x, y \in \mathbb{R}^d$ with $|\sigma^{-1}(x-y)| > \eta$, $p \geq 1$ and $t > t_0$, we have

$$\begin{aligned} \mathbb{E}|\sigma^{-1}(X_t - Y_t)|^p &= \mathbb{E}[\mathbb{E}^{(X_{T_\eta}, Y_{T_\eta})} |\sigma^{-1}(X_{t-T_\eta} - Y_{t-T_\eta})|^p] \\ &\leq C_9 \mathbb{E}[|\sigma^{-1}(X_{T_\eta} - Y_{T_\eta})| e^{-\lambda(t-T_\eta)}] \\ &\leq C_9 \eta e^{\lambda t_0} e^{-\lambda t}, \end{aligned}$$

where in the first inequality we have used (3.4), and the last inequality follows from (3.8). In particular, we have for all $|\sigma^{-1}(x-y)| > \eta$ and $t > t_0$,

$$\mathbb{E}|X_t - Y_t|^p \leq C_{10} e^{-\lambda t}$$

and so

$$W_p(\delta_x P_t, \delta_y P_t) \leq C_{10} e^{-\lambda t}.$$

Combining with all conclusions above, we complete the proof of Theorem 1.4. \square

Finally, we present the

Proof of Proposition 1.5. By the definition of κ , for all $x, y \in \mathbb{R}^d$ with $|\sigma^{-1}(x-y)| = r_0$, we have

$$\frac{\langle \sigma^{-1}(x-y), \sigma^{-1}(b(x) - b(y)) \rangle}{|\sigma^{-1}(x-y)|} \leq -2c. \quad (3.9)$$

For any fixed $x, y \in \mathbb{R}^d$ with $r = |\sigma^{-1}(x-y)|$ large enough, let $n_0 = [r/r_0]$ be the integer part of r/r_0 . Denote by $x_0 = x$ and $x_{n_0+1} = y$. We can find n_0 points $\{x_1, x_2, \dots, x_{n_0}\}$ on the line segment linking x to y , such that $|\sigma^{-1}(x_{i-1} - x_i)| = r_0$ for $i = 1, 2, \dots, n_0$ and $|\sigma^{-1}(x_{n_0} - x_{n_0+1})| = |\sigma^{-1}(x_{n_0} - y)| \leq r_0$. Then

$$\begin{aligned} &\frac{\langle \sigma^{-1}(x-y), \sigma^{-1}(b(x) - b(y)) \rangle}{|\sigma^{-1}(x-y)|} \\ &= \sum_{i=1}^{n_0} \frac{\langle \sigma^{-1}(x-y), \sigma^{-1}(b(x_{i-1}) - b(x_i)) \rangle}{|\sigma^{-1}(x-y)|} + \frac{\langle \sigma^{-1}(x-y), \sigma^{-1}(b(x_{n_0}) - b(y)) \rangle}{|\sigma^{-1}(x-y)|} \\ &= \sum_{i=1}^{n_0} \frac{\langle \sigma^{-1}(x_{i-1} - x_i), \sigma^{-1}(b(x_{i-1}) - b(x_i)) \rangle}{|\sigma^{-1}(x_{i-1} - x_i)|} + \frac{\langle \sigma^{-1}(x_{n_0} - y), \sigma^{-1}(b(x_{n_0}) - b(y)) \rangle}{|\sigma^{-1}(x_{n_0} - y)|}. \end{aligned}$$

By (3.9) and our assumption on b ,

$$\frac{\langle \sigma^{-1}(x-y), \sigma^{-1}(b(x) - b(y)) \rangle}{|\sigma^{-1}(x-y)|} \leq -2cn_0 + \delta_0.$$

Next, since $r/r_0 \leq n_0 + 1$, we have

$$-2c \frac{r}{r_0} \geq -2cn_0 - 2c$$

which implies $-2cn_0 \leq 2c - 2cr/r_0$. Therefore

$$\frac{\langle \sigma^{-1}(x - y), \sigma^{-1}(b(x) - b(y)) \rangle}{|\sigma^{-1}(x - y)|} \leq \delta_0 + 2c - \frac{2c}{r_0}r$$

for all $x, y \in \mathbb{R}^d$ with $r = |\sigma^{-1}(x - y)|$. As a result, the definition of $\kappa(r)$ leads to

$$\kappa(r) \leq \frac{1}{2}\delta_0 + c - \frac{c}{r_0}r \leq -\frac{c}{2r_0}r$$

for all $r \geq r_0(\delta_0 + 2c)/c$, and so (1.5) holds with the new constant $c/2r_0$. \square

4 Proofs of Examples and Corollaries

Proof of Example 1.6. (1) Since $\sigma = \text{Id}$, the supremum in the definition of $\kappa(r)$ is taken over all $x, y \in \mathbb{R}^d$ with $|x - y| = r$. Thus, to verify $\kappa(r) \geq 0$ for $r > 0$ large enough, it suffices to show that the supremum is nonnegative when x, y are restricted on one of the coordinate axes with $r = |x - y|$ large enough, that is, we can assume the dimension is 1. Then

$$V(x) = -(1 + x^2)^{\delta/2}, \quad \delta \in (0, 1), x \in \mathbb{R}.$$

Now the result follows immediately from the fact that $V'(x)$ is strictly increasing when $|x|$ is large enough. Indeed, we have

$$V''(x) = \delta(1 + x^2)^{\frac{\delta}{2}-2}[(1 - \delta)x^2 - 1]$$

which is positive if $|x| \geq (1 - \delta)^{-1/2}$.

On the other hand, it is easy to see that with the choices of σ and b above, the semi-group $(P_t)_{t \geq 0}$ is symmetric with respect to the probability measure $\mu(dx) = \frac{1}{Z_V} e^{V(x)} dx$. Then, according to (the proof of) Corollary 1.10 below, we know that $\mu(dx)$ fulfills the Poincaré inequality (1.15) if (1.7) is satisfied with $p = 1$; however, this is impossible, see e.g. [27, Example 4.3.1 (3)].

(2) In this case, $V(x) = -(1 + |x|^2)^{\delta/2}$ with $\delta \in [1, 2)$. First, we prove that V is strictly concave on \mathbb{R}^d . Indeed, for all $1 \leq i, j \leq d$,

$$\frac{\partial^2 V}{\partial x_i \partial x_j}(x) = \delta(1 + |x|^2)^{\delta/2-2}[(2 - \delta)x_i x_j - \delta_{ij}(1 + |x|^2)], \quad x \in \mathbb{R}^d.$$

Therefore, for any $z \in \mathbb{R}^d$ with $z \neq 0$,

$$\begin{aligned} \sum_{i,j=1}^d \frac{\partial^2 V}{\partial x_i \partial x_j}(x) z_i z_j &= \delta(1 + |x|^2)^{\delta/2-2} \sum_{i,j=1}^d [(2 - \delta)x_i x_j - \delta_{ij}(1 + |x|^2)] z_i z_j \\ &= \delta(1 + |x|^2)^{\delta/2-2} [(2 - \delta)\langle x, z \rangle^2 - (1 + |x|^2)|z|^2] \\ &\leq -\delta(1 + |x|^2)^{\delta/2-2}|z|^2 < 0, \end{aligned}$$

which implies $V(x)$ is strictly concave. Hence $\kappa(r) \leq 0$ for all $r \geq 0$. On the other hand, to show that $\kappa(r) \geq 0$ for all $r \geq 0$, as in the proof of (1), we simply look at the one dimensional case:

$$V(x) = -(1 + x^2)^{\delta/2}, \quad \delta \in [1, 2), x \in \mathbb{R}.$$

For any fixed $r > 0$ and for any $x > 0$,

$$\begin{aligned}\kappa(r) &\geq \frac{1}{2}(V'(x+r) - V'(x)) \\ &\geq \frac{r}{2} \inf_{x \leq s \leq x+r} V''(s) \\ &= \frac{r}{2} \inf_{x \leq s \leq x+r} \frac{-\delta[(\delta-1)s^2 + 1]}{(1+s^2)^{2-\delta/2}},\end{aligned}$$

which implies that

$$\kappa(r) \geq \frac{r}{2} \lim_{x \rightarrow \infty} \inf_{x \leq s \leq x+r} \frac{-\delta[(\delta-1)s^2 + 1]}{(1+s^2)^{2-\delta/2}} = 0.$$

Therefore, $\kappa(r) = 0$ for all $r > 0$.

We have seen from above that for $V(x) = -(1+|x|^2)^{\delta/2}$ with $\delta \in [1, 2)$, $\kappa(r) = 0$ for all $r \geq 0$, thus for any $x, y \in \mathbb{R}^d$,

$$\langle b(x) - b(y), x - y \rangle \leq 0.$$

Then, the assertion (1.11) immediately follows from (the proof of) Theorem 1.1, by simply using the synchronous coupling, see e.g. [2, p.2432].

Finally we prove the algebraic convergence rate (1.12). For this, we mainly follow from [9, Section 5] or [5, Section 7.2] (see also [21, Theorem 1.1]). By (2.3),

$$dr_t \leq 2 dW_t, \quad t < T,$$

where T is the coupling time of the coupling process $(X_t, Y_t)_{t \geq 0}$, and $(W_t)_{t \geq 0}$ is the same one-dimensional Brownian motion as in (2.2). Hence,

$$r_t \leq |x - y| + 2W_t, \quad t < T.$$

Let

$$\tau_z := \inf\{t > 0 : W_t = z\}.$$

Then

$$T \leq \tau_{-|x-y|/2}.$$

Denote by $W_t^* = \inf_{0 \leq s \leq t} W_s$ which has the same law as that of $-|W_t|$. Thus for any $t > 0$,

$$\begin{aligned}\mathbb{P}(r_t > 0) &= \mathbb{P}(T > t) \leq \mathbb{P}(\tau_{-|x-y|/2} > t) \\ &= \mathbb{P}(W_t^* > -|x-y|/2) = \mathbb{P}(-|W_t| > -|x-y|/2) \\ &= 2 \int_0^{|x-y|/2} \frac{1}{\sqrt{2\pi t}} e^{-s^2/2t} ds \\ &\leq \frac{|x-y|}{\sqrt{2\pi t}}.\end{aligned}$$

Therefore, for any $f \in C_b(\mathbb{R}^d)$ with $\|f\|_\infty \leq 1$, we have

$$\begin{aligned} |\mathbb{E}(f(X_t) - f(Y_t))| &= |\mathbb{E}[(f(X_t) - f(Y_t))\mathbf{1}_{\{r_t > 0\}}]| \\ &\leq 2\mathbb{P}(r_t > 0) \leq \sqrt{\frac{2}{\pi t}}|x - y|. \end{aligned}$$

In particular, by the definition of total variation distance,

$$d_{TV}(\delta_x P_t, \delta_y P_t) = \sup \{|\mathbb{E}(f(X_t) - f(Y_t))| : f \in C_b(\mathbb{R}^d), \|f\|_\infty \leq 1\} \leq \sqrt{\frac{2}{\pi t}}|x - y|.$$

The proof is complete. \square

Finally we present the proofs of the two corollaries of Theorem 1.3.

Proof of Corollary 1.8. Recall that for all $p \in [1, \infty)$, $\mathcal{P}_p(\mathbb{R}^d)$ is the space of all probability measures ν on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ satisfying $\int |z|^p \nu(dz) < \infty$. Note that, we assume the solution $(X_t)_{t \geq 0}$ to (1.1) has finite moments of all orders. In particular, for any $x \in \mathbb{R}^d$, $t > 0$ and $p \geq 1$, $\delta_x P_t \in \mathcal{P}_p(\mathbb{R}^d)$. According to (1.6) and Theorem 1.3, there is a constant $\lambda > 0$ such that for any $p \in [1, \infty)$ and any $\nu_1, \nu_2 \in \mathcal{P}_p(\mathbb{R}^d)$,

$$W_p(\nu_1 P_t, \nu_2 P_t) \leq C_p e^{-\lambda t} W_{\phi_p}(\nu_1, \nu_2), \quad t > 0, \quad (4.1)$$

where C_p is a positive constant. In particular, for any $\nu_1, \nu_2 \in \mathcal{P}_1(\mathbb{R}^d)$,

$$W_1(\nu_1 P_t, \nu_2 P_t) \leq C_1 e^{-\lambda t} W_1(\nu_1, \nu_2), \quad t > 0.$$

Let $t_0 > 0$ such that $C_1 e^{-\lambda t_0} < 1$. Then, the map $\nu \mapsto \nu P_{t_0}$ is a contraction on the complete metric space $(\mathcal{P}_1(\mathbb{R}^d), W_1)$. Hence, by the Banach fixed point theorem, there exists a unique probability measure μ_{t_0} such that $\mu_{t_0} P_{t_0} = \mu_{t_0}$. Let $\mu := t_0^{-1} \int_0^{t_0} \mu_{t_0} P_s ds$. It is easy to see that $\mu P_t = \mu$ for all $t \in [0, t_0]$ and so for all $t \in [0, \infty)$. Therefore, μ is an invariant probability for the semigroup $(P_t)_{t \geq 0}$. Moreover, for any $\nu \in \mathcal{P}_1(\mathbb{R}^d)$ and $t > 0$,

$$W_1(\nu P_t, \mu) = W_1(\nu P_t, \mu P_t) \leq C_1 e^{-\lambda t} W_1(\nu, \mu).$$

The inequality above also yields the uniqueness of the invariant measure.

On the other hand, since b is locally bounded and satisfies (1.6), it follows from [20, Theorem 4.3 (ii)] that the unique invariant measure $\mu \in \cap_{p \geq 1} \mathcal{P}_p(\mathbb{R}^d)$. Now, replacing ν_2 with μ in (4.1), we arrive at

$$W_p(\nu_1 P_t, \mu) = W_p(\nu_1 P_t, \mu P_t) \leq C_p e^{-\lambda t} W_{\phi_p}(\nu_1, \mu), \quad t > 0$$

for any $\nu_1 \in \mathcal{P}_p(\mathbb{R}^d)$. The proof is completed. \square

Proof of Corollary 1.10. Let $(P_t)_{t \geq 0}$ be the semigroup generated by $L = \Delta - \nabla U \cdot \nabla$. Then $(P_t)_{t \geq 0}$ is symmetric with respect to the probability measure μ . Since the Hessian matrix $\text{Hess}(U) \geq -K$, we deduce that $\kappa(r) \leq Kr/2$ for all $r > 0$. Moreover, replacing b by $-\nabla U$ in the definition of $\kappa(r)$, we deduce from (1.14) that $\kappa(L) < 0$. According to

Proposition 1.5 and Theorem 1.3, we know that there exist two positive constants C, λ such that for all $t > 0$ and $x, y \in \mathbb{R}^d$,

$$W_1(\delta_x P_t, \delta_y P_t) \leq C e^{-\lambda t} |x - y|.$$

This implies that (e.g. see [6, Theorem 5.10] or [24, Theorem 5.10])

$$\|P_t f\|_{\text{Lip}} \leq C e^{-\lambda t} \|f\|_{\text{Lip}} \quad (4.2)$$

holds for any $t > 0$ and any Lipschitz continuous function f , where $\|f\|_{\text{Lip}}$ denotes the Lipschitz semi-norm with respect to the Euclidean norm $|\cdot|$.

On the other hand, for any $f \in C_c^2(\mathbb{R}^d)$, by (4.2), we have

$$\begin{aligned} \text{Var}_\mu(f) &= \mu(f^2) - \mu(f)^2 = - \int_{\mathbb{R}^d} \int_0^\infty \partial_t (P_t f)^2 dt d\mu \\ &= -2 \int_0^\infty \int_{\mathbb{R}^d} P_t f L P_t f d\mu dt \\ &= \int_0^\infty \int_{\mathbb{R}^d} |\nabla P_t f|^2 d\mu dt \\ &\leq \int_0^\infty \|P_t f\|_{\text{Lip}}^2 dt \leq \frac{C^2}{2\lambda} \|f\|_{\text{Lip}}^2. \end{aligned}$$

Replacing f with $P_t f$ in the equality above, we arrive at

$$\text{Var}_\mu(P_t f) \leq \frac{C^2}{2\lambda} \|P_t f\|_{\text{Lip}}^2 \leq \frac{C^4 e^{-2\lambda t}}{2\lambda} \|f\|_{\text{Lip}}^2.$$

Next, we follow the proof of [23, Lemma 2.2] to show that the inequality above yields the desired Poincaré inequality. Indeed, for every f with $\mu(f) = 0$ and $\mu(f^2) = 1$. By the spectral representation theorem, we have

$$\begin{aligned} \|P_t f\|_{L^2(\mu)}^2 &= \int_0^\infty e^{-2ut} d(E_u f, f) \\ &\geq \left[\int_0^\infty e^{-2us} d(E_u f, f) \right]^{t/s} \\ &= \|P_s f\|_{L^2(\mu)}^{2t/s}, \quad t \geq s, \end{aligned}$$

where in the inequality above we have used Jensen's inequality. Thus,

$$\|P_s f\|_{L^2(\mu)}^2 \leq \left[\frac{C^4 e^{-2\lambda t}}{2\lambda} \|f\|_{\text{Lip}}^2 \right]^{s/t} \leq \frac{C^{4s/t}}{(2\lambda)^{s/t}} \|f\|_{\text{Lip}}^{2s/t} e^{-2\lambda s}.$$

Letting $t \rightarrow \infty$, we get that

$$\|P_s f\|_{L^2(\mu)}^2 \leq e^{-2\lambda s},$$

which is equivalent to the desired Poincaré inequality, see e.g. [27, Theorem 1.1.1]. \square

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